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Left Generalized Derivations on Prime Γ-Rings

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Abstract. Let M be a prime Γ -ring with 2-torsion free, I a nonzero ideal of M and $f: M \to M$ a left generalized derivation of M, with associated nonzero derivation d on M. If $f(x) \in Z(M)$ for all $x \in I$, then M is a commutative Γ-ring.

Keywords: Gamma ring, prime gamma ring, derivation, generalized derivation, left generalized derivation, commutators.

AMS Mathematics Subject Classification (2010): 06F25

1. Introduction

The notion of Γ-ring was first introduction by Nobusawa [9] and also shown that Γ-ring, more general than rings. Barnes [1] slightly weakened the conditions in the definitions of a Γ-rings in the sense of Nobusawa. After the study of Γ-rings by Nobusawa [9] and Barnes [1], many researchers have a done lot of work and have obtained some generalizations of the corresponding results in ring theory [6][8]. Barnes [1] and kyuno [8] studied the structure of Γ -ring and obtained various generalizations of the corresponding results of ring theory. Hvala [4] introduced the concept of Generalized derivations in rings. Dey, Paul and Rakhimov [3] discussed some properties of Generalized derivations in semiprime gamma rings Bresar [2] studied on the distance of the composition of two derivations to the generalized derivations. Jaya Subba Reddy. et al. [5] studied centralizing and commutating left generalized derivation on prime ring is commutative. Jaya Subba Reddy et al. [12] studied some results of symmetric reverse bi-derivations on prime rings, Ozturk et al. [10] studied on derivations of prime gamma rings. Khan et al. [6,7] studied on derivations and generalized derivations on prime Γ-rings is a commutative. In this paper we extended some results on left generalized derivations on prime Γ-ring is a commutative.

2. Preliminaries

If M and Γ are additive abelian groups and there exists a mapping $M \times \Gamma \times M \to M$ which satisfies the following conditions:
For all $a, b \in M$ and $\alpha, \beta \in \Gamma$,

(i) (a, β, b), denoted by $a\beta b$, is an element of M
(ii) $(a+b)\beta c = a\beta c + b\beta c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\beta(b+c) = a\beta b + a\beta c$
(iii) $(a\alpha b)\beta c = a\alpha(b\beta c)$
then M is called a Γ-ring [1]. It is known that from (i), (iii) the following follows:

$$0\beta b = a0b = a\beta 0 = 0 \quad \text{(A)}$$

for all a and b in M and all β in Γ [1].

Every ring is a Γ-ring with $M = \Gamma$. However a Γ-ring need not be a ring. Let M be a Γ-ring, then M is called a prime Γ-ring, if $a\Gamma M\Gamma b = 0$ implies $a = 0$ or $b = 0$, for all $a, b \in M$ and M is called a semiprime Γ-ring, if $a\Gamma M\Gamma a = 0$ implies $a = 0$, for all $a \in M$. Every prime Γ-ring is obviously semiprime. If M is a Γ-ring, then M is said to be 2-torsion free if $2x = 0$ implies $x = 0$, for all $x \in M$. An additive subgroup I of M is called a left (right) ideal of M if $M\Gamma I \subseteq I$ ($I\Gamma M \subseteq I$). If I is both left and right ideal of M, then we say I is an ideal of M. Moreover, the set $Z(M) = \{x \in M: x\beta y = y\beta x \ \ \forall \beta \in \Gamma, \ y \in M\}$ is called the centre of the Γ-ring M. We shall write $[x, y]_\beta = x\beta y - y\beta x$, for all $x, y \in M$ and $\beta \in \Gamma$. We shall make use of the basic commutator identities: $[x\beta y, z]_\alpha = [x, z]_\alpha \beta y + x\beta[y, z]_\alpha$ and $[x, y\beta z]_\alpha = [x, y]_\alpha \beta z + y\beta[x, z]_\alpha$, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. If Γ-ring satisfies the assumption (B) $a\alpha b\beta c = a\beta b\alpha c$, for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. Let M be a Γ-ring. An additive mapping $d: M \to M$ is called a derivation on M if $d(x\gamma y) = d(x)\gamma y + x\gamma d(y)$ holds for all $x, y \in M$ and $\gamma \in \Gamma$. An additive mapping $f: M \to M$ is called a generalized derivation if there exists a derivation $d: M \to M$ such that $f(x\gamma y) = f(x)\gamma y + x\gamma d(y)$ holds for all $x, y \in M$ and $\gamma \in \Gamma$. An additive mapping $f: M \to M$ is called a left generalized derivation if there exists a derivation $d: M \to M$ such that $f(x\gamma y) = x\gamma f(y) + d(x)\gamma y$ holds for all $x, y \in M$ and $\gamma \in \Gamma$. A derivation of the form $x \to a\alpha x + x\alpha b$ where a, b are fixed elements of M and $\alpha \in \Gamma$ is called generalized inner derivation. An additive mapping $T: M \to M$ is called a left (right) centralizer if $T(x\alpha y) = T(x)\alpha y$ ($T(x\alpha y) = x\alpha T(y)$) for all $x, y \in M$ and $\alpha \in \Gamma$.

Lemma 2.1. Let M be a prime Γ-ring with 2-torsion free and I a nonzero ideal of M. Let $f: M \to M$ be a left generalized derivation of M, associated with derivation d. If $f(y) = 0$, for all $y \in I$, then $f = 0$.

Proof: For all $x, y \in I$ and $\beta \in \Gamma$, $f(x\beta y) = 0$. That is, $x\beta f(y) + d(x)\beta y = 0$, which implies $d(x)\beta y = 0$. Let $z \in M, \alpha \in \Gamma$. The last relation along with (A) gives, $d(x)\beta z\alpha y = 0$. Since M is prime Γ-ring and I is a nonzero ideal, so $d(x) = 0$, for all $x \in I$. Hence , by hypothesis, $f(x\beta r) = 0$, for all $x \in I$, and $\beta \in \Gamma$, and $r \in M$. That is, $x\beta f(r) + d(x)\beta r = 0$, which gives $x\beta f(r) = 0$. Let $w \in M, \gamma \in \Gamma$. The last relation along with (A), implies $x\gamma w\beta f(r) = 0$. Since I is nonzero and primeness of M, gives $f = 0$.

Lemma 2.2. Let I be a nonzero ideal of a prime Γ-ring M, $a \in M$ and $f \neq 0$ is a left generalized derivation of M, with associated nonzero derivation d, then
(i) If $f(y)\beta a = 0$ for all $y \in I$ and $\beta \in \Gamma$, then $a = 0$,
(ii)If $a\beta f(y) = 0$ for all $y \in I$ and $\beta \in \Gamma$, then $a = 0$.
Proof:

(i) For any $y \in I, r \in M$ and $\alpha, \beta \in \Gamma$, $f(r\alpha y)\beta a = 0$. That is,
$r\alpha f(y)\beta a + d(r)\alpha y\beta a = 0$ Which implies, $d(r)\alpha y\beta a = 0$. Since I is a nonzero ideal of M and $d \neq 0$, we get $a = 0$.

(ii) Proof is similar to (i).

3. Main results

Theorem 3.1. Let M be a prime **Γ**-ring with 2-torsion free and I a nonzero ideal of M. Let $f: M \to M$ be a left generalized derivation of M, with associated nonzero derivation d on M. If $f(x) \in Z(M)$ for all $x \in I$, then M is a commutative **Γ**-ring.

Proof: Using hypothesis, we have $[f(x\beta y), x]_\alpha = 0$, for all $x, y \in I$, $\alpha, \beta \in \Gamma$, which gives $[x\beta f(y) + d(x)\beta y, x]_\alpha = 0$

$$x\beta[f(y), x]_\alpha + [x, x]_\alpha \beta f(y) + d(x)\beta[y, x]_\alpha + [d(x), x]_\alpha \beta y = 0$$

Using hypothesis, we get

$d(x)\beta[y, x]_\alpha + [d(x), x]_\alpha \beta y = 0$

$d(x)\beta y\alpha x - d(x)\beta x\alpha y + d(x)\alpha x\beta y - x\alpha d(x)\beta y = 0$

Using (B), from the last equation we get

$d(x)\beta y\alpha x - d(x)\alpha x\beta y + d(x)\alpha x\beta y - x\alpha d(x)\beta y = 0$

$d(x)\beta y\alpha x - x\alpha d(x)\beta y = 0$, for all $, y \in I$, $\alpha, \beta \in \Gamma$. (1)

Let $z \in I$. Replacing y by $z\beta y$ in equation (1), we get

$d(x)\beta z\beta y\alpha x - x\alpha d(x)\beta z\beta y = 0$

Which along with equation (1) and (B) gives ,

$d(x)\beta z\beta y\alpha x - d(x)\beta z\beta x\alpha y = 0$

$d(x)\beta z\beta[y, x]_\alpha = 0$, for all $x, y \in I$ and $\alpha, \beta \in \Gamma$.

Since I is a nonzero ideal of M and $d \neq 0$, therefore M is a commutative **Γ**- ring.

Theorem 3.2. Let M be a prime **Γ**-ring with 2-torsion free and I a nonzero ideal of M. Let $f: M \to M$ be a generalized derivation and left generalized derivation of M, with associated derivation d on M. If $a \in M$ and $[f(x), a]_\alpha = 0$, for all $x \in I$, $\alpha \in \Gamma$, then either $a \in Z(M)$ or $d(a) = 0$.

Proof: Using hypothesis, we have

$[f(x\beta y), a]_\alpha = 0$, for any $x \in M$, $y \in I$ and $\alpha, \beta \in \Gamma$.

This gives $[x\beta d(y) + f(x)\beta y, a]_\alpha = 0$

$[x\beta d(y), a]_\alpha + [f(x)\beta y, a]_\alpha = 0$

The last equation gives

$x\beta[d(y), a]_\alpha + [x, a]_\alpha \beta d(y) + f(x)\beta[y, a]_\alpha + [f(x), a]_\alpha \beta y = 0$

Using hypothesis, from the last equation we get

$x\beta[d(y), a]_\alpha + [x, a]_\alpha \beta d(y) + f(x)\beta[y, a]_\alpha = 0$

$x\beta d(y)\alpha a - x\beta a\alpha d(y) + x\alpha a\beta d(y) - a\alpha x\beta d(y) + f(x)\beta y\alpha a - f(x)\beta a\alpha y = 0$

Using (B), from the last equation we get

$x\beta d(y)\alpha a - x\alpha a\beta d(y) + x\alpha a\beta d(y) - a\alpha x\beta d(y) + f(x)\beta y\alpha a - f(x)\beta a\alpha y = 0$

$x\beta d(y)\alpha a - a\alpha x\beta d(y) + f(x)\beta y\alpha a - f(x)\beta a\alpha y = 0$ (2)

Let $z \in M$. Replacing x by $z\gamma x$ in equation (2), we get

$z\gamma x\beta d(y)\alpha a - a\alpha z\gamma x\beta d(y) + f(z\gamma x)\beta y\alpha a - f(z\gamma x)\beta a\alpha y = 0$

$$z\gamma x\beta d(y)\alpha a - a\alpha z\gamma x\beta d(y) + z\gamma(f(x)\beta y\alpha a - f(x)\beta a\alpha y) + d(z)\gamma x\beta(y\alpha a - a\alpha y) = 0$$

Using equation (2), from the last equation we get

$z\gamma x\beta d(y)\alpha a - a\alpha z\gamma x\beta d(y) + z\gamma(a\alpha x\beta d(y) - x\beta d(y)\alpha a) + d(z)\gamma x\beta[y,a]_\alpha = 0$
$z\gamma x\beta d(y)\alpha a - a\alpha z\gamma x\beta d(y) + z\gamma a\alpha x\beta d(y) - z\gamma x\beta d(y)\alpha a + d(z)\gamma x\beta[y,a]_\alpha = 0$
Using (B), from the last equation we get
$-a\alpha z\gamma x\beta d(y) + z\alpha a\gamma x\beta d(y) + d(z)\gamma x\beta[y,a]_\alpha = 0$
$[z,a]_\alpha\gamma x\beta d(y) + d(z)\gamma x\beta[y,a]_\alpha = 0$
Replacing y by a from the last equation, we get
$[z,a]_\alpha\gamma x\beta d(a) + d(z)\gamma x\beta[a,a]_\alpha = 0$
$[z,a]_\alpha\gamma x\beta d(a) = 0$, for all $x \in I$, $z \in M$ and $\alpha,\beta,\gamma \in \Gamma$.
Since I is nonzero ideal of prime Γ-ring M, therefore either $d(a) = 0$ or $a \in Z(M)$.

Corollary 3.2.1. Let M be a prime Γ-ring with 2-torsion free and I a nonzero ideal of M. Let $f: M \to M$ be a left generalized derivation of M, with associated derivation d on M. If $[f(x), f(y)]_\beta = 0$, for all $x, y \in I$, $\beta \in \Gamma$, then M is a commutative Γ-ring.
Proof: Using Theorem 3.2, we have $f(I) \subset Z(M)$, we get the corollary 3.2.1 proof.

Theorem 3.3. Let M be a prime Γ-ring with 2-torsion free and I a nonzero ideal of M. Let $f: M \to M$ be a left generalized derivation of M, with associated derivation d on M. If $f(x\beta y) = f(x)\beta f(y)$, for all $x, y \in I$, $\beta \in \Gamma$, then $d = 0$.
Proof: $f(x\beta y) = x\beta f(y) + d(x)\beta y$, for any $x, y \in I$, $\beta \in \Gamma$.
$f(x)\beta f(y) = x\beta f(y) + d(x)\beta y$ (3)
Let $w \in I$, $\gamma \in \Gamma$.Then replacing y by $w\gamma y$ in equation (3), we get
$f(x)\beta f(w\gamma y) = x\beta f(w\gamma y) + d(x)\beta w\gamma y$
$d(x)\beta w\gamma(f(y) - y) = 0$, for all $x, y \in I$, and $\gamma, \beta \in \Gamma$.
Since I is a nonzero ideal of the prime Γ-ring M, therefore either $f(y) - y = 0$ for all $y \in I$ or $d(x) = 0$ for all $x \in I$. If $f(y) - y = 0$, then $f(y) = y$ for all $y \in I$. Replacing y by $y\beta x$ in the last equation, we get $f(y\beta x) = y\beta x$, which implies $y\beta f(x) + d(y)\beta x = y\beta x$, which gives $y\beta x + d(y)\beta x = y\beta x$. That is $d(y)\beta x = 0$, for all $x, y \in I$, $\beta \in \Gamma$. Thus $d(y) = 0$ for all $y \in I$ for both cases. So $d = 0$.

Theorem 3.4. Let M be a prime Γ-ring with 2-torsion free and I a nonzero ideal of M. Let $f: M \to M$ be a left generalized derivation of M, with associated derivation d on M. If $f(x\beta y) = f(y)\beta f(x)$, for all $x, y \in I$, $\beta \in \Gamma$, then $d = 0$.
Proof: $f(x\beta y) = x\beta f(y) + d(x)\beta y$, for all $x, y \in I$, $\beta \in \Gamma$.
$f(y)\beta f(x) = x\beta f(y) + d(x)\beta y$ (4)
Let $x \in I$, $\gamma \in \Gamma$. Replacing y by $x\gamma y$ in equation (4), we get
$f(x\gamma y)\beta f(x) = x\beta f(x\gamma y) + d(x)\beta x\gamma y$
$x\gamma f(y)\beta f(x) + d(x)\gamma y\beta f(x) = x\beta f(y)\gamma f(x) + d(x)\beta x\gamma y$
Using (B), from the last equation we get
$x\beta f(y)\gamma f(x) + d(x)\gamma y\beta f(x) = x\beta f(y)\gamma f(x) + d(x)\beta x\gamma y$
$d(x)\gamma y\beta f(x) = d(x)\beta x\gamma y$ (5)
Let $w \in I$, $\alpha \in \Gamma$. Then replacing y by $y\alpha w$, we get
$d(x)\gamma y\alpha w\beta f(x) = d(x)\beta x\gamma y\alpha w$
Using equation (5) in above equation, we get
$d(x)\gamma y\alpha w\beta f(x) = d(x)\gamma y\beta f(x)\alpha w$
Using (B), from the last equation we get

$d(x)\gamma y\alpha f(x)\beta w - d(x)\gamma y\alpha w\beta f(x) = 0$

$$d(x)\gamma y\alpha[f(x), w]_\beta = 0$$

Since I is a nonzero ideal of the prime Γ-ring M. Therefore either $d(x) = 0$ for all $x \in I$ or $[f(x), w]_\beta = 0$ for all $x, w \in I$ and $\beta \in \Gamma$. Let $A = \{x \in I: d(x) = 0\}$ and $B = \{x \in I: [f(x), w]_\beta = 0, \forall\, w \in I\}$. Obviously A and B are additive subgroups of I. Moreover I is the set theoretic union of A and B. But a group cannot be set theoretic union of two proper sub groups . Hence either $A = I$ or $B = I$. If $A = I$, we have $d(R) = 0$, which completes the proof. If $B = I$, then $0 = [f(x), w]_\beta = w\alpha[f(x), r]_\beta$ for all $x, w \in I$, $r \in M$ and $\alpha, \beta \in \Gamma$. Thus, we obtain $f(I) \subset Z(M)$, Using Theorem 3.2, we get $d = 0$.

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